Multifractal image denoising

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Abstract

We present a new method for image denoising based on singularity analysis. The image is characterized via its multifractal spectrum, which mode yields the most frequent Hölder exponent. Using 2-microlocal analysis, we define an operator that shifts the spectrum so that the transformed image has almost sure Hölder exponent a little above 2. This manipulation leads to a smooth image while preserving the relative strength of the singularities (as, for instance, edges or textures) in the signal. Experimental results on Radar images are presented.

Keywords: multifractal analysis, denoising, 2-microlocalisation, wavelets, Radar imaging.

1 Introduction

There has been a tremendous amount of work dedicated to image restoration [2, 5, 4, 3] in recent years. In classical methods, one aims at filtering the noisy data, in order to obtain a signal which is as close as possible (in a predefined sense) to the unknown original one by making assumptions on the noise model. However, the perturbation of a signal with a noise generally induces modifications in its regularity. More refined methods [3] allow to control the smoothness of the restored image. Recently, an approach based on wavelet coefficients shrinkage [2, 4] has been proposed that insures that the filtered image will belong to the same *global* regularity space as the original one. We adopt here a slightly different approach: we make no attempt to model the noise but try instead to refine the regularity analysis in order to obtain a *pointwise* control of the signal smoothness. Loosely speaking, the multifractal spectrum will replace the Besov indexes used in [2] as a characterization of the signal regularity.

We recall in Section 2 the basic ideas of the multifractal analysis of images. Section 3 describes the principle and implementation of our denoising method. Some results on Radar images are given in Section 4. Finally, the Appendix reviews some notions of 2-microlocal analysis useful in the denoising operator design.

2 Multifractal analysis of images

Multifractal analysis has recently drawn much attention as a tool for studying the structure of singular signals, both in theory and in applications [7, 10, 1, 9].

In the multifractal scheme, the pointwise structure of a singular signal is analyzed through the so-called "multifractal spectrum", which gives either geometrical or probabilistic information about the distribution of points having the same singularity. The former characterization is obtained through the *Hausdorff spectrum* f_h, while the latter is given by the *large deviation spectrum* f_g. More precisely, $f_h(\alpha)$ is the Hausdorff dimension of the points which Hölder exponent equals α, while $f_g(\alpha)$ evaluates the convergence speed of the probability that coarse grained exponents at resolution ϵ equals α when ϵ tends to 0.

The "multifractal formalism" assesses that, in some situations, f_g and f_h coincide. Its interest is twofold: first, it gives a way to evaluate f_h, which is typically much more difficult to compute than f_g. Second, it allows to draw several consequences of practical interest on the signal. For instance, a smooth contour in an image is characterized both by a geometrical dimension of one, and by a given probability to be hit if one draws a point at random in the image.

In order to simplify the discussion, we will only deal here with f_h. Formally, in the one dimensional case, the Hölder exponent of a signal f at x_0 is:

$$\alpha(x_0) = \sup\{\tilde{\alpha} \,:\, \exists\, P_m, \exists\, C > 0\, / \\ |f(x) - P_m(x - x_0)| < C|x - x_0|^{\tilde{\alpha}}\} \tag{1}$$

where P_m is a polynomial of degree not exceeding the integer part of $\tilde{\alpha}$ and x is in a neighbourhood of x_0. The *Hausdorff multifractal spectrum* is then defined as:

$$f_h(\alpha) = dim_H E_(\alpha) = dim_H \{x|\alpha(x) = \alpha\} \tag{2}$$

where dim_H denotes the *Hausdorff dimension.* [9]

The multifractal analysis of images [8] consists in defining a function on the image, computing its multifractal spectrum, and classifying each point according to the corresponding value of $(\alpha, f_h(\alpha))$ in a geometric fashion. The value of α gives a *local* information about the pointwise regularity: for instance an ideal step edge point in an image without noise is characterized by a given value. The value of $f_h(\alpha)$ yields a *global* information: a point on a smooth contour belongs to a set E_α whose dimension is 1, a point contained in a homogeneous region has $f_h(\alpha) = 2$, etc ... In fact, we may define the type of a point (i.e. edge, corner, smooth region ...) through its associated $f_h(\alpha)$ value: for $t \in [0, 2]$, x is called a point of type t if $f_h(\alpha(x)) = t$.

3 Multifractal denoising

3.1 Principle

The basic idea of multifractal denoising is best explained on a simple example. The aim is to get rid of "insignificant" irregularities while keeping "meaningful" singularities. Moreover, after denoising, "most" points should lie in smooth regions. On figure 1 all points have the same regularity $\alpha = 0.5$ except $t = 1$ where a step occurs ($\alpha = 0$). In term of multifractal spectrum, $f_h(0.5) = 1$, $f_h(0) = 0$ and $f_h(\alpha) = -\infty$ for $\alpha \notin \{0, 0.5\}$. We want to find a simple method to obtain a shifted spectrum $\tilde{f}_h$ so that the maximum of $\tilde{f}_h$ is located at $\alpha = 1 + \epsilon$, $\epsilon > 0$. This will imply that the transformed signal is almost everywhere (*a.e.*) smooth while preserving the "rare" event "step at 1". More formally, our assumption regarding the presence of generic non significant singularities is modeled as: (for a 2-D signal)

$$\exists\,!\; \alpha_0 < 1 \,:\, f_h(\alpha_0) = 2 \tag{3}$$

We then apply a transform operator O so that the spectrum $\tilde{f}_h$ of the modified image is

$$\tilde{f}_h(\alpha) = f_h(\alpha + \alpha_0 - \epsilon - 1) \tag{4}$$

The resulting image will be such that "most" points will have Hölder exponents a little above 1 and will consequently be *a.e.* smooth. On the other hand, since the shape of the spectrum is preserved by O, the relative strength of the singularities is not modified, i.e. salient visual features remains unchanged. In addition, such a procedure does not induce, in principle, any loss of information. We describe in the following section a practical method for deriving O. A result of applying O to the signal on figure 1 is shown on figure 2.

3.2 Operator design

In the following we will use a decomposition of the signal on a wavelet basis $\{\Psi_(j, k), \Phi_k\}$ of sufficient regularity. We will denote by (c_k^j) the wavelet coefficients.

We recall that, loosely speaking, if f has Hölder exponent α at x_0, the (c_k^j) in $D_{x_0} = \{(j, k) \,:\, x_0 \in$

$supp(\Psi_{j,k})\}$ decrease as $O(2^{-j\alpha})$ when j tends to infinity[11, 7]. In order to modify the Hölder regularity, a natural idea is to act upon the wavelet coefficients. The simplest way to increase the regularity from α to α' is to multiply c_k^j by $2^{-j(\Delta\alpha)}$ (where $\Delta\alpha = \alpha' - \alpha$). However, such a naive approach does not insure that transformed signal will have Hölder exponent α'. To obtain this result, it is convenient to consider a finer characterization of regularity, namely 2-microlocal analysis. Notations and details are described in the Appendix.

2-microlocal analysis explains why, when the (c_k^j) are multiplied by $2^{-j(\Delta\alpha)}$, the regularity gain can be larger than $\Delta\alpha$, resulting in a multifractal spectrum distortion. For instance, in the case of the chirp $C(x) = |x|^\alpha .sin(\frac{1}{|x|^\beta})$, this gain is $\Delta\alpha(1+\beta)$.

Let us call $(\nabla_{x_0}^\theta)^{-1}$ the operator which consists in multiplying each c_k^j in D_{x_0} by $2^{-j\theta}$. Let:

$$f^\theta = (\nabla_{x_0}^\theta)^{-1}(f) \tag{5}$$

By (11) and (12), one has:

$$S_{(f^\theta,x_0)}(s) = S_{(f,x_0)}(s-\theta) \tag{6}$$

and the Hölder exponent of f^θ at x_0 is:

$$\begin{aligned} \alpha(f^\theta) &= \sup\{s : s + S_{(f^\theta,x_0)}(s) = 0\} \\ &= \sup\{s+\theta : s + S_{(f,x_0)}(s) = \theta\} \end{aligned} \tag{7}$$

Let:

$$\tau_{(f,x_0)}(\theta) = \sup\{s : s + S_{(f,x_0)}(s) = \theta\} \tag{8}$$

Then:

$$\alpha(f^\theta) = \tau_{(f,x_0)}(\theta) + \theta,\ (\theta > 0). \tag{9}$$

One can then prove the following lemma [6]:

Lemma 3.1 *Suppose $f \in C_{x_0}^\alpha$, $\alpha > 0$, then:*

1. *$\tau(\theta)$ is non-decreasing in $]0,\infty[$.*
2. *$\tau(0) = \alpha$.*
3. *$\tau(\theta)$ is continuous in $]0,\infty[$.*

Properties *1.* and *2.* come from (15) and (14) while *3.* comes from the convexity of the 2-microlocal domain. This leads to:

Proposition 3.1 *[6] $\forall \alpha_\star > \alpha : \exists!\, \theta_\star$:*

$$f^{\theta_\star} \in C_{x_0}^{\alpha_\star - \epsilon},\ \forall\epsilon > 0 \tag{10}$$

This results means that, for any $\Delta\alpha > 0$, there is a unique θ such that the operator $(\nabla_{x_0}^\theta)^{-1}$ increases the pointwise regularity by $\Delta\alpha$. Lemma 3.1 insures the numerical stability of the inverse problem, i.e. finding θ given $\Delta\alpha$

Lack of space prevents us from describing the practical algorithm for computing $S(s)$ and the details of the operator implementation (See [6]).

4 Application

We show in this section an application of the method explained above to the denoising a SAR image. Noise is difficult to model on such signals, because it is far from being additive and Gaussian. Correlated K distributions have been proposed in this case [12], for which it is not an easy task to design an analytic denoising method.

We also display results where the multifractal spectrum has not been shifted but linearly distorded (dilated). Again, the ditorsion coefficient might be defined so that the spectrum maximum is obtain for a regularity exponent a little above 1.

The original images are displayed on figure 3 and 7. On the first one, we applied 3 different regularizations: a Kuan filtering and our multifractal shifting and linear distorsion methods. The Kuan filtering is visually efficient because it preserves a good spatial definition but might change the noise nature instead of attenuating it. On the contrary, the multifractal spectrum shifting seems to slightly oversmooth small regions while significantly reducing the noise on homogoneous regions. One can see that the sea region for instance, has been greatly smoothed while most edges are preserved. The multifractal spectrum distorsion seems to be a good trade-off between the two preceeding methods, yielding a smooth image which nevertheless exhibits sharp edges.

In the second SAR image, we present, in addition to the Kuan filtering and the multifractal spectrum shift and distorsion, a denoising with the wavelet shrinkage method.

As a conclusion, our method seems to yield results comparable to previously known ones. Improvements need to be done on the estimation on the $S_{(f,x_0)}$ function and on various algorithmic details. The algorithm can also be modified in order to perform more general "filterings" of the multifractal spectrum. A first step in this direction has been presented through the multifractal linear distorsion scheme.

An electronic version of the images can be found at the following adress:
http://www-rocq.inria.fr/fractales/Publications/MultDenSCIA97.ps.gz

Appendix: 2-microlocal analysis

Definition

We will consider here a definition that involves only conditions on the signal wavelet coefficients.

Definition 1 *A distribution f is said to belong to the 2-microlocal space $C^{s,s'}_{x_0}(\mathbb{R}^n)$ if: $\exists C > 0 : \forall \in \mathbb{N}, \forall k \in Z$*

$$|c^j_k| \leq C 2^{-(\frac{n}{2}+s)j}(1+|k-2^j x_0|)^{-s'} \qquad (11)$$

$$|< f, \phi_k >| \leq C(1+|k-x_0|)^{-s'} \qquad (12)$$

Where ϕ is the scaling function of the wavelet basis.

Properties

These regularity spaces are stable under the action of differential and pseudo-differential operators. Our main interest in 2-microlocal spaces comes from their relationship with the usual local Hölder spaces.

Theorem 1 *[7, 11]*

$$f \in C^{s,s'}_{x_0} \Leftrightarrow \frac{df}{dx} \in C^{s-1,s'}_{x_0} \qquad (13)$$

$$(s+s') > 0, f \in C^{s,s'}_{x_0} \Rightarrow f \in C^{s}_{x_0} \qquad (14)$$

$$f \in C^{s}_{x_0} \Rightarrow f \in C^{s,-s}_{x_0} \qquad (15)$$

For a given function, the set of all possible couples (s, s') at a given point is a region of $\mathbb{R}^2$ and not merely a half line as it is the case with the Hölder exponents. Let us call this region the 2-microlocal domain of f at x_0. Considering (11) and (12) one easily checks that:

$$f \in C^{s,s'}_{x_0} \Rightarrow f \in C^{S,S'}_{x_0} \qquad (16)$$

$$\forall S \leq s, \forall S + S' \leq s + s' \qquad (17)$$

Suppose that, for a given s, there exists an s_0 such that $f \in C^{s,s'_0}_{x_0}$. Then, for this s, we can define the value:

$$S_{(f,x_0)}(s) = sup\{s' \in \mathbb{R} : f \in C^{s,s'}_{x_0}\} \qquad (18)$$

Thus, the knowledge of the application $S_{(f,x_0)}(s)$ defined on the domain $\Gamma_{(f,x_0)}$ of all admissible s is sufficient to characterize completely the whole 2-microlocal domain of f at x_0. A classical example is the chirp $C(x) = |x|^\alpha.sin(\frac{1}{|x|^\beta})$, for which

$$S_{(C,0)}(s) = \frac{\alpha}{\beta} - s\frac{1+\beta}{\beta} \qquad (19)$$

$$\Gamma_{(C,0)} = \mathbb{R} \qquad (20)$$

One easily checks that $S_{(f,x_0)}$ is always a concave function.

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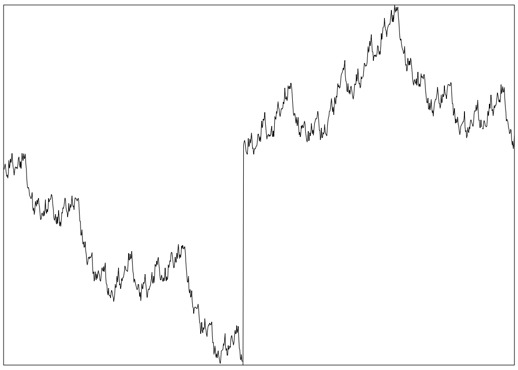

Figure 1: A signal which Hölder exponent is 0.5 every where except in 0.5 where it is 0.

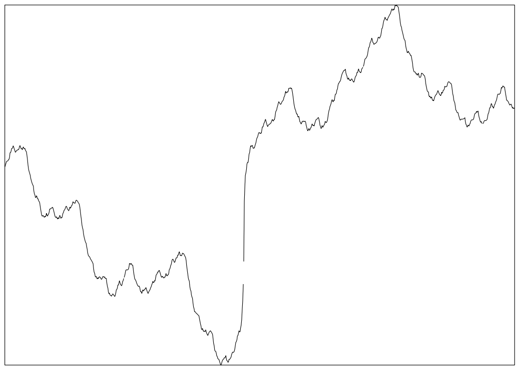

Figure 2: The signal in figure 1 which multifractal spectrum has been shifted by 0.5

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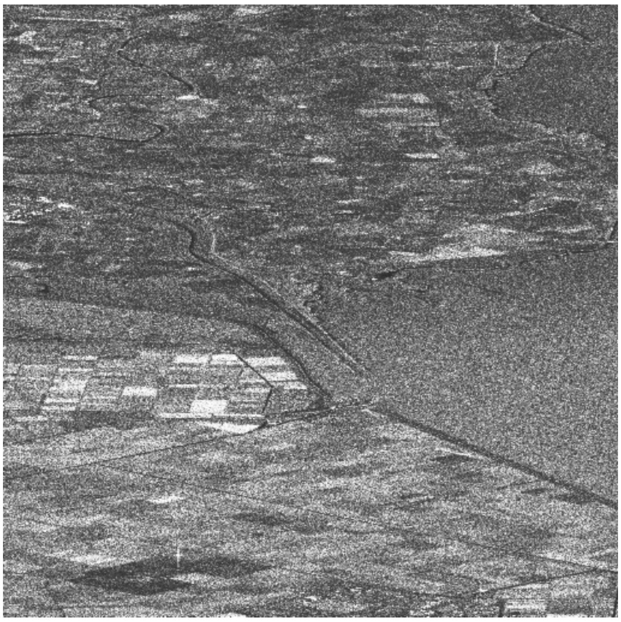

Figure 3: The original SAR Radar Image

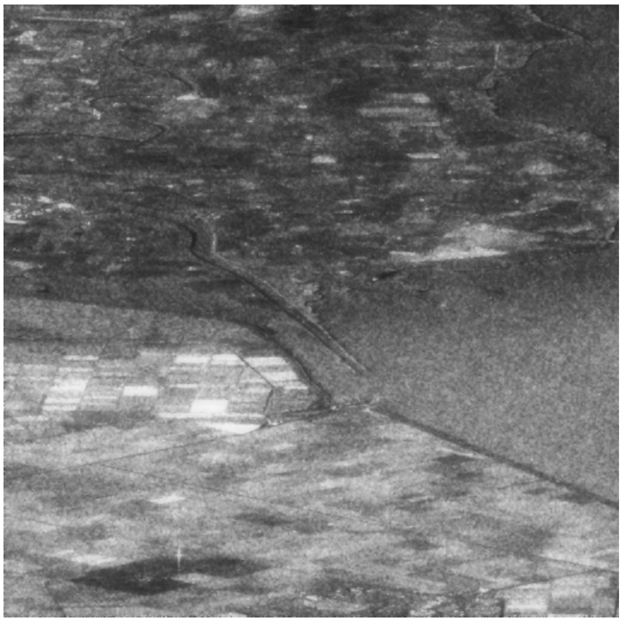

Figure 5: Denoising by multifractal spectrum shifting

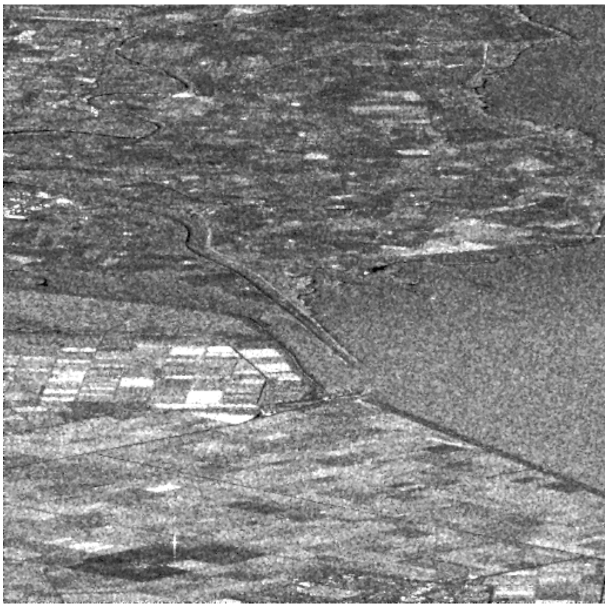

Figure 4: Image regularized by a Kuan filtering

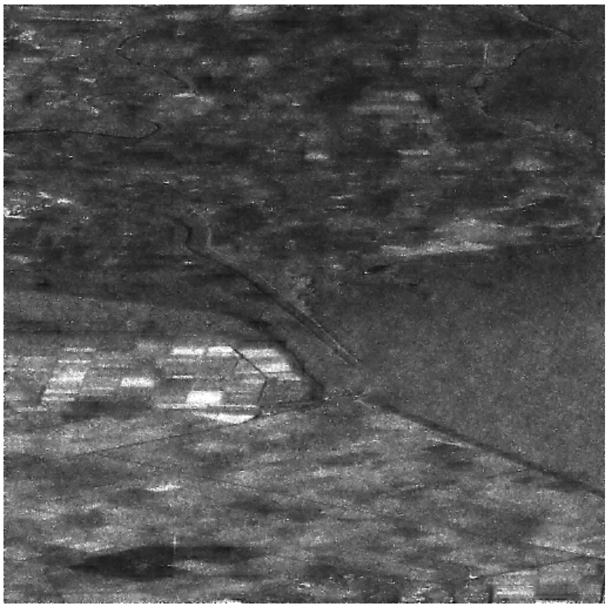

Figure 6: Denoising by multifractal spectrum distortion

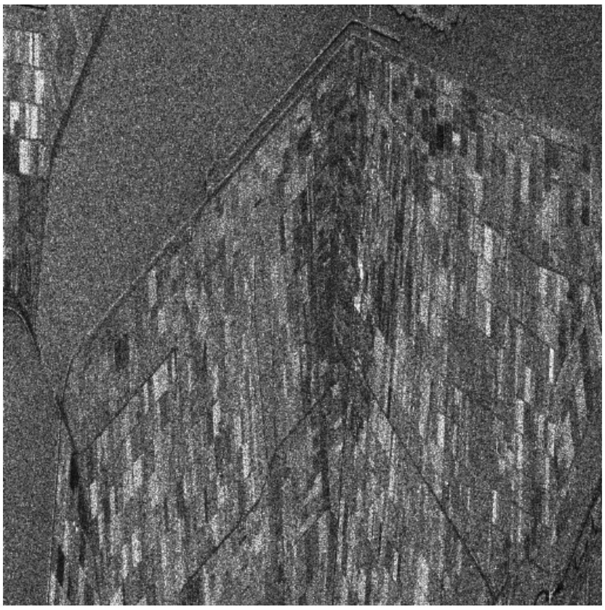

Figure 7: The original SAR Radar Image

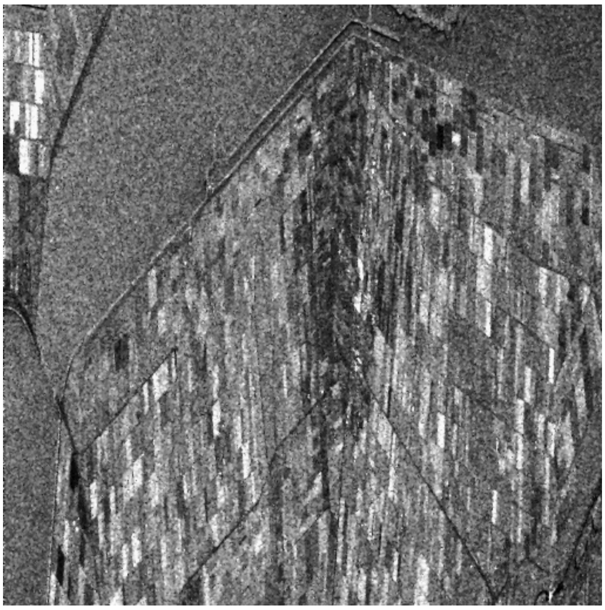

Figure 8: Image regularized by a Kuan filtering

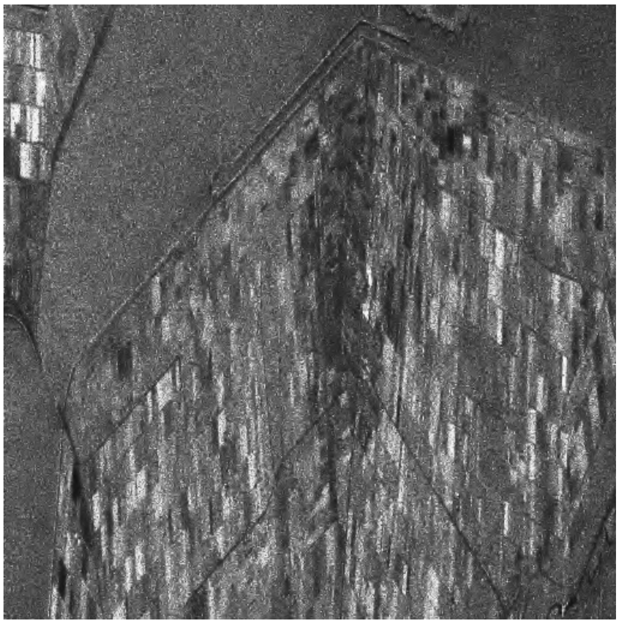

Figure 9: Denoising by wavelet shrinkage

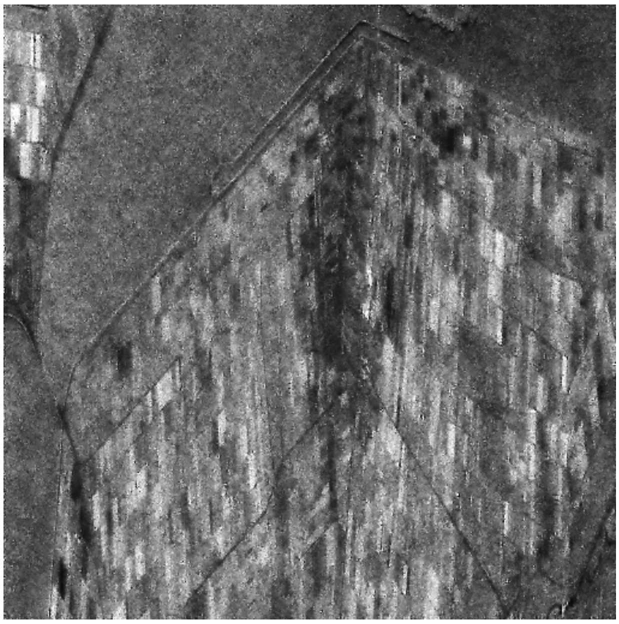

Figure 10: Denoising by multifractal spectrum distorsion

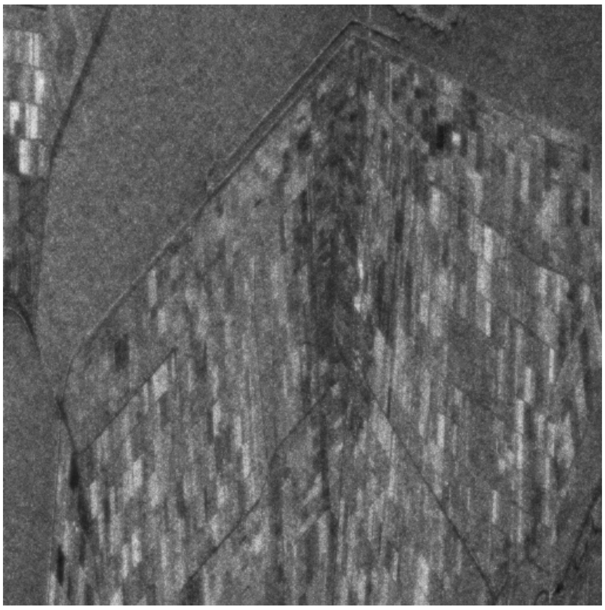

Figure 11: Denoising by multifractal spectrum shifting